Joe Dinius PHYS 515B
HW2 November 27, 2008

1. Show that it is impossible for an electron to absorb or emit a photon.

   Solution. First consider the process of an electron absorbing a photon. WLOG, assume electron moves along x-direction and the photon moves along y-direction.

   Conservation of momentum requires

   $$(P_e^\alpha + P_{p_\alpha})^2 = P_e^{'\alpha} P'_{e_\alpha}$$

   where subscripts e and p represent the electron and proton, respectively. Computing the vector products yields

   $$\begin{aligned}(1+\beta_1^2)E_e^2 + 2E_p^2 + 2E_eE_p &= E_e^{'2}(1+\beta_1^{'2}+\beta_2^{'2}) \\ \text{(require energy be conserved)} &= (E_e+E_p)^2(1+\beta_1^{'2}+\beta_2^{'2}) \\ &= (1+\beta_1^{'2}+\beta_2^{'2})E_e^2 + (1+\beta_1^{'2}+\beta_2^{'2})E_p^2 + (1+\beta_1^{'2}+\beta_2^{'2})2E_eE_p.\end{aligned}$$

   Equality requires

   $$\begin{aligned}(1+\beta_1^{'2}+\beta_2^{'2}) &= (1+\beta_1^2) \\ &= 1 \\ &= 2,\end{aligned}$$

   which is clearly a contradiction.

   The same argument can be made, leading to the same result, for the emission of a photon.

   Therefore, a free electron can neither absorb nor emit a photon.

2. Jackson 11.23 (a):
   In a collision process a particle of mass m_2, at rest in the laboratory, is struck by a particle of mass m_1, momentum $\mathbf{p}_{lab}$ and total energy E_{lab}. In the collision the two initial particles are transformed into two others of mass m_3 and m_4. The configurations of the momentum vectors in the center of momentum (cm) frame (traditionally called the center-of-mass frame) and the laboratory frame are shown in the figure.

   (a) Use invariant scalar products to show that the total energy W in the cm frame has its square given by

   $$W^2 = m_1^2 + m_2^2 + 2m_2E_{lab}$$

   and that the cms 3-momentum $\mathbf{p}'$ is

   $$\mathbf{p}' = \frac{m_2\mathbf{p}_{lab}}{W}.$$

Solution. Let

$$\begin{aligned} P_1^\alpha &= (E_{lab}, \mathbf{p}),\ P_1'^\alpha = (E_1', \mathbf{p}' = \mathbf{p}_1') \\ P_2^\alpha &= (m_2, \mathbf{0}),\ P_2'^\alpha = (E_2', -\mathbf{p}' = \mathbf{p}_2'). \end{aligned}$$

Energy and momentum conservation in lab frame gives

$$P_1^\alpha + P_2^\alpha = P_3'^\alpha + P_4'^\alpha.$$

The total center-of-mass energy is W is

$$\begin{aligned} W^2 &= (E_1' + E_2')^2 \\ &= (E_1' + E_2')^2 - (\mathbf{p}_1' + \mathbf{p}_2')^2 \\ &= (P_1'^\alpha + P_2'^\alpha)^2. \end{aligned}$$

This last term is Lorentz invariant, so

$$\begin{aligned} W^2 = (P_1'^\alpha + P_2'^\alpha)^2 &= (P_1^\alpha + P_2^\alpha)^2 \\ &= P_1^\alpha P_{1_\alpha} + P_2^\alpha P_{2_\alpha} + 2P_1^\alpha P_{2_\alpha} \\ &= m_1^2 + m_2^2 + 2m_2 E_1 \end{aligned}$$

Now to find **p**', look at

$$\begin{aligned} (P_1^\alpha P_{2_\alpha})^2 &= (m_2 E_1)^2 = m_2^2(p_{lab}^2 + m_1^2) = m_2^2 p_{lab}^2 + m_1^2 m_2^2 \\ (P_1'^\alpha P_{2_\alpha}')^2 &= (E_1' E_2' + p'^2)^2 = E_1'^2 E_2'^2 + 2E_1' E_2' p'^2 + p'^4 \end{aligned}$$

where p without bold represents the magnitude of the different vectors.

Consider the second of the two above results,

$$\begin{aligned} (P_1'^\alpha P_{2_\alpha}')^2 &= (p'^2 + m_1^2)(p'^2 + m_2^2) + 2E_1' E_2' p'^2 + p'^4 \\ &= 2p'^4 + (m_1^2 + m_2^2 + 2E_1' E_2')p'^2 + m_1^2 m_2^2 \\ &= p'^2(E_1'^2 + 2E_1' E_2' + E_2'^2) + m_1^2 m_2^2 \\ &= p'^2 W^2 + m_1^2 m_2^2. \end{aligned}$$

Considering Lorentz invariance again, we get

$$\begin{aligned} m_2^2 p_{lab}^2 &= p'^2 W^2 \\ p' &= \frac{m_2}{W} p_{lab}. \end{aligned}$$

But, $\mathbf{p}'$ is in the same direction as $\mathbf{p}_{lab}$, so

$$\mathbf{p}' = \frac{m_2}{W} \mathbf{p}_{lab}.$$

3. A mirror is moving through vacuum with relativistic speed v in the x direction. A beam of light with frequency ω_i is normally incident from $x+\ +\infty$.

(a) What is the frequency of the reflected light?

Solution. For a photon, $p = (\frac{\hbar\omega}{c}, \hbar\vec{\omega})$. Consider Σ is the rest frame of light source and observer and Σ' the rest frame of mirror. The frequency transforms from Σ to Σ' via a Lorentz transformation,

$$\begin{aligned}
\omega_i' &= \frac{1}{\sqrt{1-v^2/c^2}}(\omega_i - \frac{v}{c}\omega_i) \\
\omega_i &= \frac{1}{\sqrt{1-v^2/c^2}}(\omega_i' + \frac{v}{c}\omega_i').
\end{aligned}$$

(b) What is the energy of each reflected photon?

Solution. On reflection, $\omega_r' = \omega_i'$. So for the observer in Σ,

$$\begin{aligned}
\omega_r &= \frac{1}{\sqrt{1-v^2/c^2}}(\omega_r' + \frac{v}{c}\omega_r') \\
&= \frac{1}{\sqrt{1-v^2/c^2}}\left(1+\frac{v}{c}\right)^2\omega_i \\
&= \left(\frac{c+v}{c-v}\right)\omega_i.
\end{aligned}$$

(c) The average energy flux of the incident beam is P_i. What is the average reflected energy flux?

Solution. If n is the number of photons per unit volume of the beam, its average energy flux is $n\hbar\omega$. The average energy flux of the reflected beam is

$$\begin{aligned}
P_r &= nc\hbar\omega_r \\
&= \left(\frac{c+v}{c-v}\right)nc\hbar\omega_i \\
&= \left(\frac{c+v}{c-v}\right)P_i.
\end{aligned}$$